**Foundations of Deep Learning – Homework Assignment #2**Adi Album & Tomer Epshtein

**Part 3: (1)**

Prove (Lemma): (“matricization of outer product = Kronecker product of matricizations”)

Let and be tensors of order and respectively. Let and denote by the set obtained by subtracting from each element in . Then:

Proof:

Let be a tensor of order and be a tensor of order .

Denote where .

**Are the two sides of the target equation equal?**

1. Sanity check – they have equal dimensions… Both sides of the equation are matrices of dimensions:
2. We’ll use linearity to make our proof less complicated.
   1. Linearity of tensor outer product:  
      For any scalars and any tensors such that and have equal dimensions.
   2. Linearity of Matricization:  
      For any scalars and any tensors such that and have equal dimensions.
   3. Linearity of Kronecker Product:  
      For any scalars and any tensors such that and have equal dimensions.

Any tensor can be represented as a linear combinations of tensors with scalars , such that is a tensor with ‘1’ in a single entry and ‘0’ in all other entries.

1. Claim:  
   Let be some tensor with ‘1’ in a single entry and ‘0’ in all other entries,  
   Let be some tensor with ‘1’ in a single entry and ‘0’ in all other entries.  
   Let and denote by the set obtained by subtracting from each element in .  
   Then:
2. Assume we proved Claim (3).  
   We’ll first show this completes our proof:  
     
   Let be a tensor of order and be a tensor of order .  
     
   By (2) we have and

{Linearity of tensor outer product}  
   
{Linearity of matricization}  
   
{(3)}  
   
{Linearity of keronecker product}  
   
{Linearity of matricization}

As desired.

It remains to prove claim 3:

Claim 3:  
Let be some tensor with ‘1’ in a single entry and ‘0’ in all other entries,  
Let be some tensor with ‘1’ in a single entry and ‘0’ in all other entries.  
Let and denote by the set obtained by subtracting from each element in .  
Then:

Proof of claim 3:

Let be some tensor with ‘1’ in a single entry and ‘0’ in all other entries,  
I.e. for some . And is for all other entries.

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I.e. for some . And is for all other entries.

Start by viewing LHS:

* **Tensor outer product:**  
   is a tensor defined by:  
  So this produces a tensor with ‘1’ in a single entry. The entry with index:  
  And ‘0’ in every other entry.
* **Matricization:** is a matrix  
    
  Defined such that:  
  for every and

Where:

I.e. We have a matrix with a single ‘1’ entry in index corresponding to index in .

Simply putting it, if we denote the matrix in LHS as  
We have a matrix with a single ‘1’ entry at and all other entries are ‘0’ where

Let’s move on to viewing RHS:

* **Matricizations:**
  + We have an order tensor , so its matricization w.r.t is:

Is defined such that:  
for every .  
Where:  
 with a single ‘1’ entry at and all other entries are ‘0’. Where:

* + And an order tensor , so its matricization w.r.t is:  
    Is defined such that:  
    for every , .  
    Where:

Simply putting it, this yields a matrix with a single ‘1’ entry at and all other entries are ‘0’. Where:

**Kronecker Product:**  
  
We now have two matrices:  
,   
Each matrix with a single ‘1’ entry and all other entries being ‘0’.  
’s ‘1’ entry: , ‘1’ entry: .  
  
 & ’s Kronecker product yields a matrix with a single ‘1’ entry at index where:

Bringing it **ALL** together…

* On the LHS we got a matrix :with a single ‘1’ entry at and all other entries are ‘0’ where
* On the RHS we got a matrix   
  with a single ‘1’ entry at and all other entries are ‘0’ where

Plotting and into and yields equality:

and

I.e matrix (=LHS) is equal to matrix (=RHS)